Topic 7 - Spherical Coordinates

Let Q be a point in 3-dimensions.

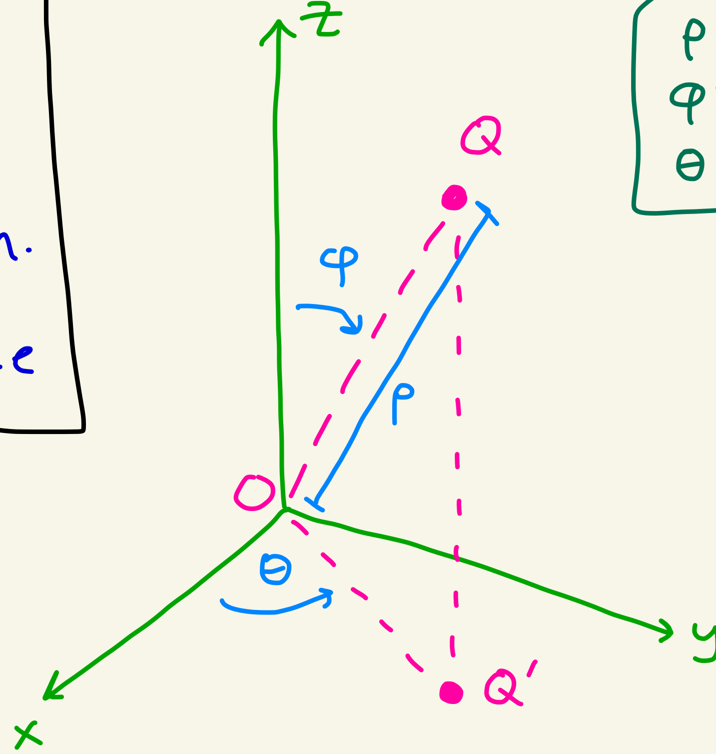
Let O be the origin.

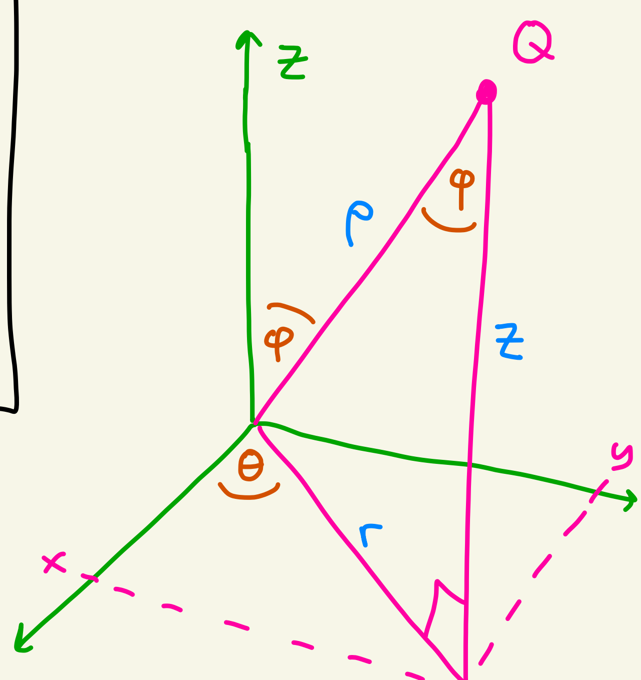
- Let ρ be the distance between O and Q
- Let φ be the angle between the positive z-axis and the line segment $\overline{OQ}$
- Let Q' be the projection of Q into the xy-plane. That is, if $Q=(x,y,z)$ then $Q'=(x,y,0)$. Let θ be the angle for the polar coordinates of Q'.

The spherical coordinates of $Q=(x,y,z)$ are:

$$\rho^2 = x^2+y^2+z^2$$
$$x = \rho\sin(\varphi)\cos(\theta)$$
$$y = \rho\sin(\varphi)\sin(\theta)$$
$$z = \rho\cos(\varphi)$$

ρ = rho
φ = phi
θ = theta

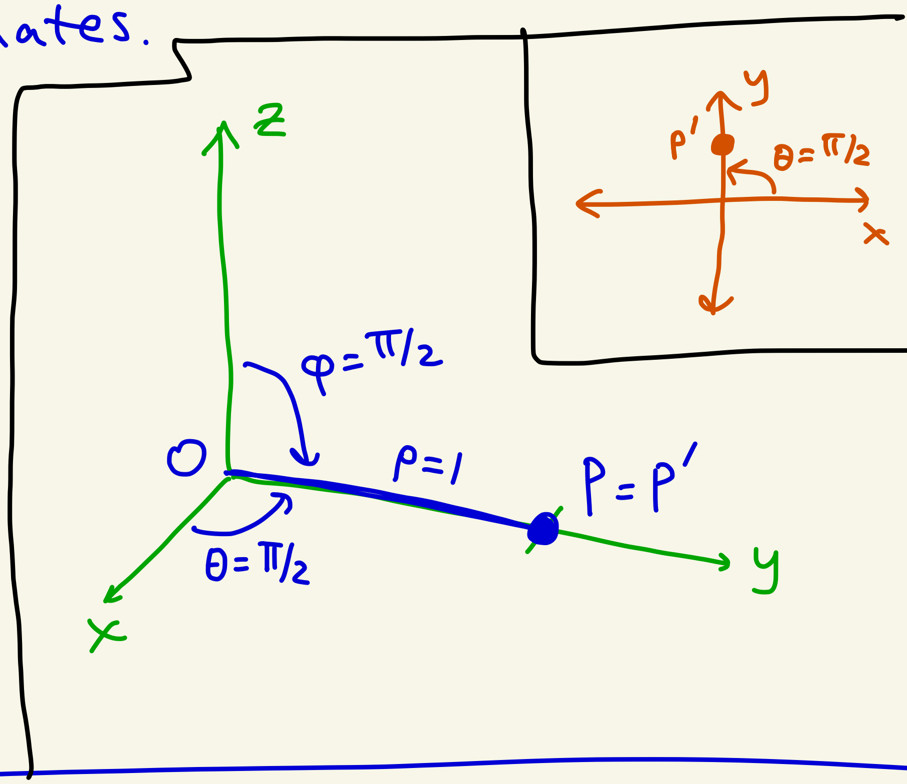




<u>calculations:</u>

$$z = \rho\cos(\varphi)$$
$$r = \rho\sin(\varphi)$$
$$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$$
$$y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$$
$$\rho^2 = r^2+z^2 = x^2+y^2+z^2$$

Ex: Convert $P = (x, y, z) = (0, 1, 0)$ into spherical coordinates.



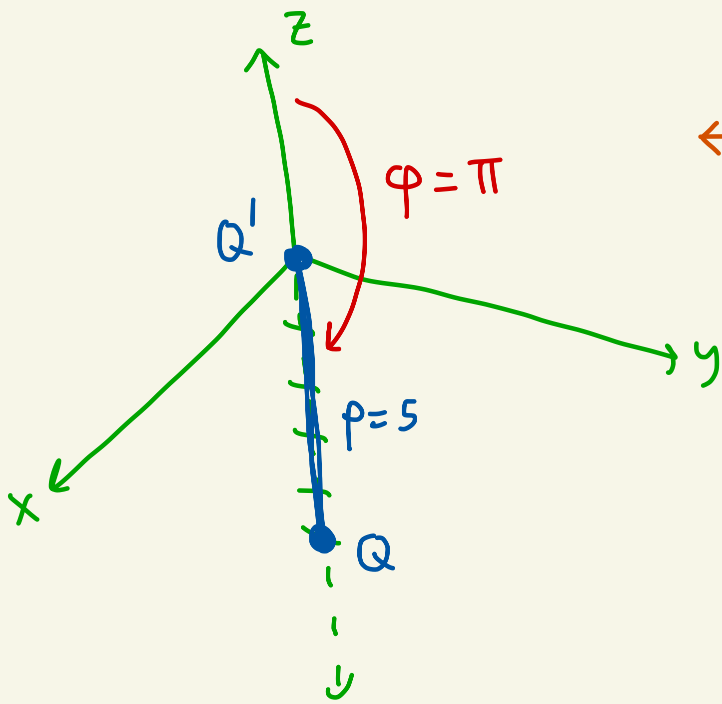
We get

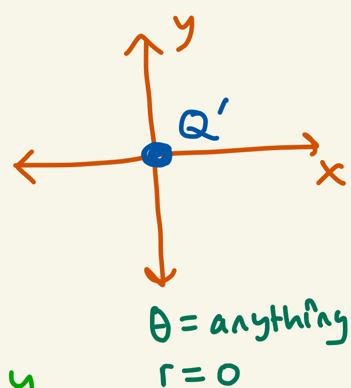
$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$= \sqrt{0^2 + 1^2 + 0^2} = 1$$

$$\varphi = \pi/2$$

$$\theta = \pi/2$$

Ex: Convert $Q = (x, y, z) = (0, 0, -5)$ into spherical coordinates.



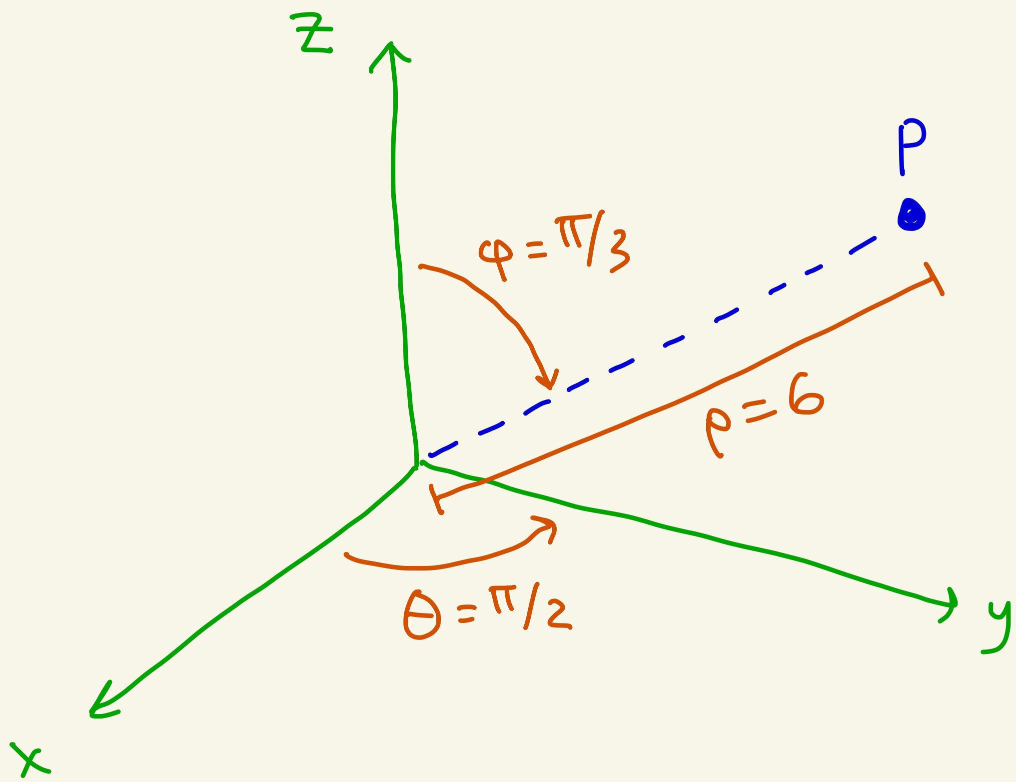


$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$= \sqrt{0^2 + 0^2 + (-5)^2}$$
$$= 5$$

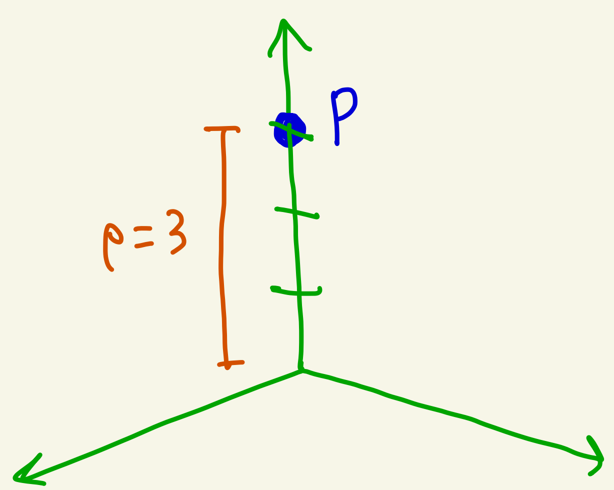
$$\varphi = \pi$$

$\theta = 0$ ← θ can be any angle

Ex: Plot the point P with spherical coordinates $(\rho, \varphi, \theta) = (6, \pi/3, \pi/2)$

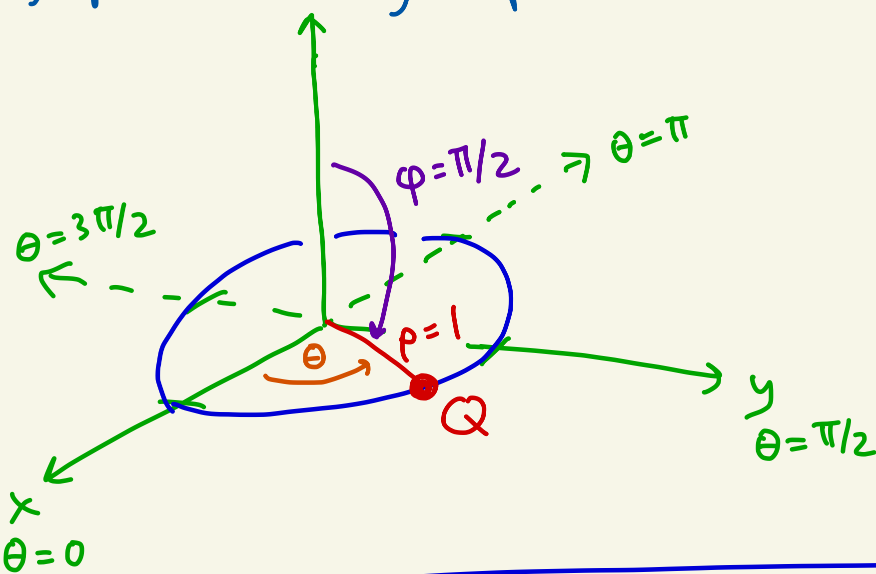


Ex: Plot the point P with spherical coordinates $(\rho, \varphi, \theta) = (3, 0, 0)$

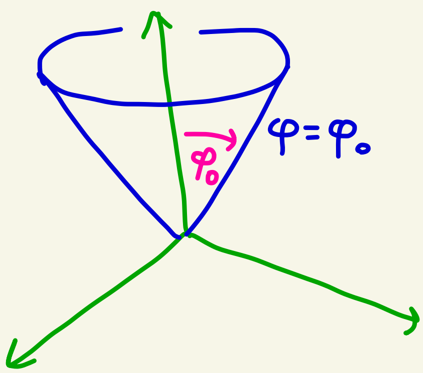


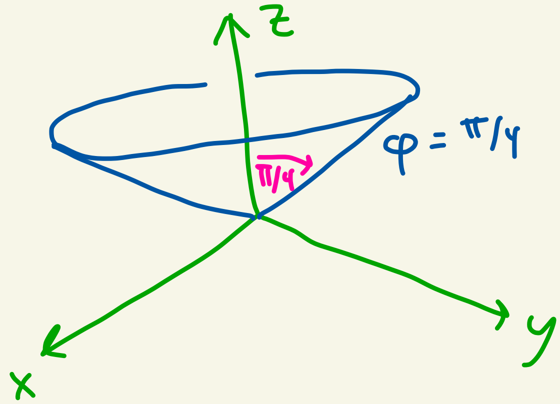
Ex: Find a description of the unit circle in the xy-plane using spherical coordinates.

$$\boxed{\begin{array}{l}\rho = 1\\ \varphi = \pi/2\\ 0 \le \theta \le 2\pi\end{array}}$$

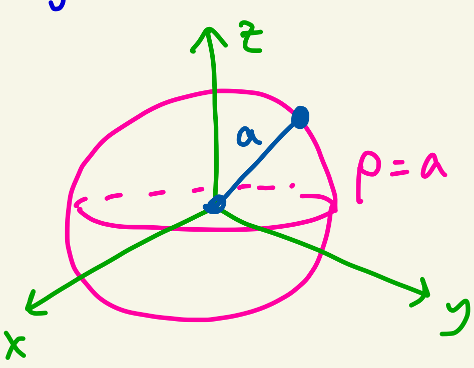


Ex: The surface given by $\varphi = \varphi_0$ is a cone.





Ex: The surface of the sphere of radius a centered at the origin is given by $\rho = a$

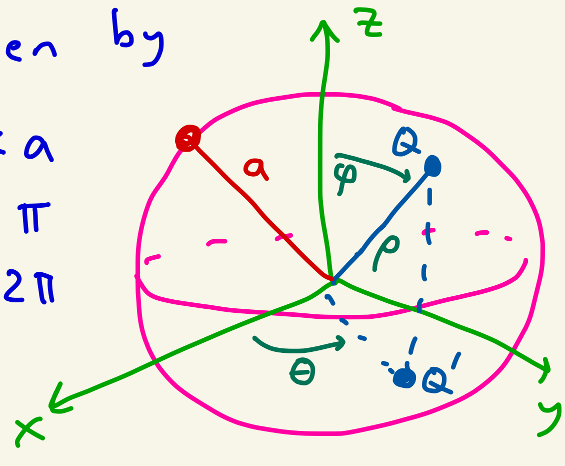


Ex: The solid inside the sphere of radius $a > 0$ centered at the origin is given by

$0 \le \rho \le a$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$



Theorem: Let E be the solid consisting of all points satisfying $a \le \rho \le b$, $c \le \varphi \le d$, $f \le \theta \le g$. Let f be continuous on E.
Then

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_f^g \int_c^d \int_a^b f\Big(\underbrace{\rho\sin(\varphi)\cos(\theta)}_{x}, \underbrace{\rho\sin(\varphi)\sin(\theta)}_{y}, \underbrace{\rho\cos(\varphi)}_{z}\Big) \cdot \underbrace{\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta}_{dV}$$

Ex: Find the volume of the sphere of radius $a > 0$.

$$\iiint_E 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$\boxed{\begin{array}{l} 0 \le \rho \le a \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le \pi \end{array}}$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{\rho^3}{3}\sin(\varphi) \Big|_{\rho=0}^{a} \right) d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{a^3}{3}\sin(\varphi)\,d\varphi\,d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \left(-\cos(\varphi) \Big|_{\varphi=0}^{\pi} \right) d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \underbrace{\left(-\cos(\pi) - (-\cos(0)) \right)}_{1+1} d\theta$$

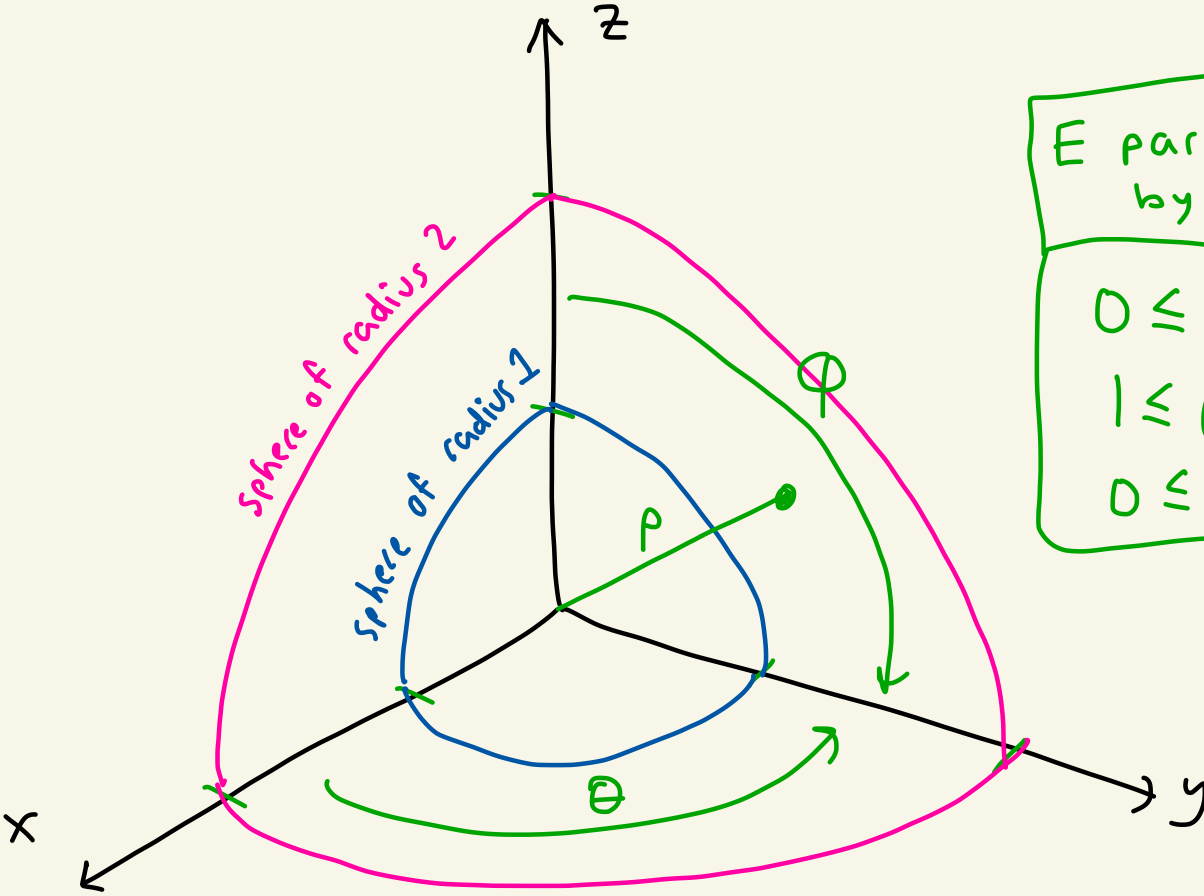
$$= \frac{a^3}{3} \int_0^{2\pi} 2 \, d\theta$$

$$= \frac{a^3}{3} \left[2\theta \Big|_0^{2\pi} \right]$$

$$= \frac{a^3}{3} \left[2 \cdot 2\pi - 2 \cdot 0 \right]$$

$$= \boxed{\frac{4\pi}{3} a^3}$$

<u>Ex:</u> Find the mass of the solid E that lies in the first octant between the spheres of radius 1 and radius 2 centered at origin, where the density function is given by $f(x,y,z) = (x^2+y^2+z^2)^{-3/2}$.



E parameterized by

$0 \leq \varphi \leq \pi/2$

$1 \leq \rho \leq 2$

$0 \leq \theta \leq \pi/2$

$$\iiint_E \underbrace{(x^2+y^2+z^2)}_{\rho^2}{}^{-3/2} \underbrace{dV}_{\rho^2 \sin(\varphi)}$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 (\rho^2)^{-3/2} \cdot \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left(\int_1^2 \frac{1}{\rho} \sin(\varphi)\, d\rho \right) d\varphi\, d\theta$$

$$\boxed{(\rho^2)^{-3/2} \rho^2 = \rho^{-3} \rho^2 = \rho^{-1}}$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left(\ln|\rho| \cdot \sin(\varphi) \Big|_{\rho=1}^{2} \right) d\varphi\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left(\underbrace{\ln|2|}_{\ln(2)} \cdot \sin(\varphi) - \underbrace{\ln|1|}_{0} \cdot \sin(\varphi) \right) d\varphi\, d\theta$$

$$= \ln(2) \int_0^{\pi/2} \int_0^{\pi/2} \sin(\varphi)\, d\varphi\, d\theta$$

$$= \ln(2) \int_0^{\pi/2} \left[-\cos(\varphi) \Big|_{\varphi=0}^{\pi/2} \right] d\theta$$

$$= \ln(2) \int_0^{\pi/2} \left[\underbrace{-\cos(\pi/2) - (-\cos(0))}_{-0-(-1)} \right] d\theta$$

$$= \ln(2) \int_0^{\pi/2} 1 \cdot d\theta$$

$$= \ln(2) \left[\theta \Big|_0^{\pi/2} \right]$$

$$= \ln(2) \left[\pi/2 - 0 \right] = \boxed{(\pi/2) \ln(2) \approx 1.08879\ldots}$$